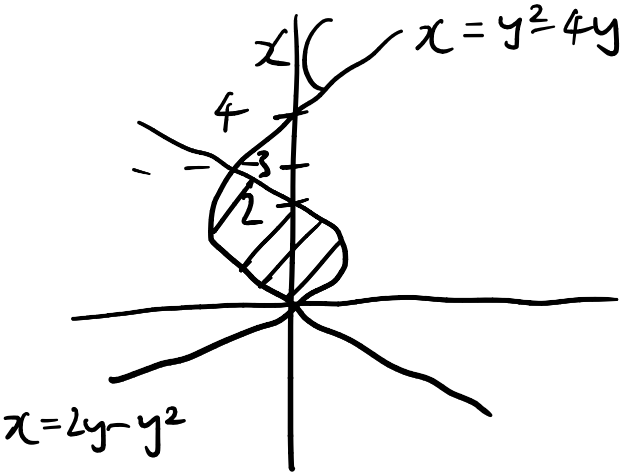
22:57–23:57 20/9/25

$3/6 = 50\%$

1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

$x = y(y-4)$, $x = y(2-y) = -y(y-2)$



$y^2 - 4y = 2y - y^2$

$2y^2 = 6y$

$y^2 - 3y = 0$

$y(y-3) = 0$

$y = 0, 3$

$$\text{Area} = \int_0^3 (2y - y^2) - (y^2 - 4y)\, dy$$

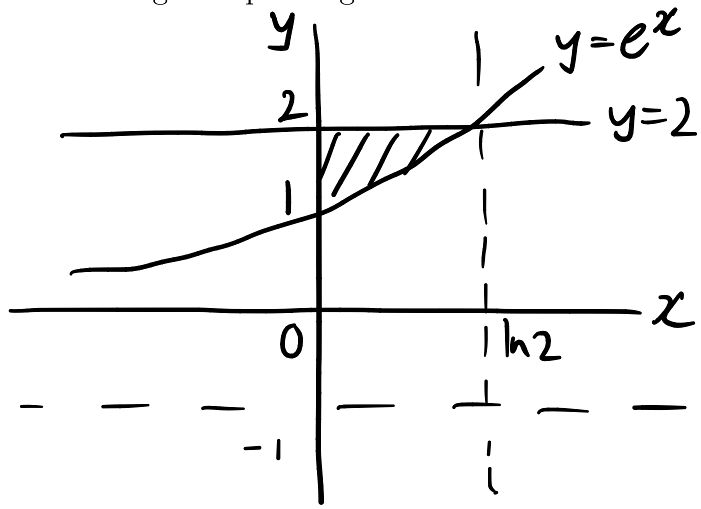
$$= \int_0^3 -2y^2 + 6y\, dy$$

$$= -\frac{2y^3}{3} + \frac{6y^2}{2} \Big|_0^3$$

$$= \left(-\frac{2}{\cancel{3}} \cdot \cancel{27}^{9} + 3 \cdot 9\right)$$

$$= 9$$

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$. You only need to give a definite integral expressing the volume. Do not solve the integral.



$e^x = 2$

$x = \ln 2$

$$V = \int_0^{\ln 2} \pi R^2 - \pi\left(3-(2-e^x)\right)^2 dx$$

$$= \int_0^{\ln 2} \pi \cdot 9 - \pi\left(9-6(2-e^x)+(2-e^x)^2\right) dx$$

$$= \int_0^{\ln 2} 6\pi(2-e^x) - \pi(2-e^x)^2 dx$$

$$\pi\left(3^2-(1+e^x)^2\right) dx$$

3. Evaluate each of the following expressions

(a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i \cdot \frac{3}{n}\right)^2 \frac{3}{n}$$

$\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{3}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} (1+i\Delta x)^2 \Delta x$

$= \int_0^3 f(x)\,dx$

$= \int_0^3 (1+x)^2\,dx$

$= \int_0^3 1+2x+x^2\,dx$

$= x + \frac{2x^2}{2} + \frac{x^3}{3} \Big|_0^3$

$= 3+9+9$

$= 21$

(b) The value $f(4)$ for the continuous function f satisfying

$$x \sin \pi x = \int_0^{x^2} f(t)\,dt$$

$\frac{d}{dx}(x\sin\pi x) = f(x^2)\cdot 2x$

$f(x^2)\cdot 2x = \sin\pi x + x\cdot\pi(\cos\pi x)$

$f(x^2) = \frac{\sin\pi x}{2x} + \frac{\pi\cos\pi x}{2}$

$\Rightarrow f(x) = \frac{\sin\pi\sqrt{x}}{2\sqrt{x}} + \frac{\pi\cos\pi\sqrt{x}}{2}$

$f(4) = 0 + \frac{\pi(1)}{2}$

$= \frac{\pi}{2}$

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density). For a plane figure with uniform density, the coordinates of the center of mass are given by weighted averages, where the weighting function is the moment of inertia:

$$\left(\frac{\int x f(x)\, dx}{\int f(x)\, dx}, \frac{\int y g(y)\, dy}{\int g(y)\, dy} \right).$$

$\bar{x} = \int x$

Area $= \frac{1}{2} rh$

(b) Pappus' Theorem says that the volume of the solid formed by rotating a region is the area of the region times the distance traveled by the rotating centroid. Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

5. Given a definite integral

$$\int_a^b f(x)\,dx,$$

let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

$$T_n = \frac{b-a}{n}\left(\frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$M_n = \frac{b-a}{n}\left(\frac{y_0+y_1}{2} + \frac{y_1+y_2}{2} + \cdots + \frac{y_{n-1}+y_n}{2}\right)$$

$$= \frac{b-a}{n}\left(\frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$S_n = \frac{b-a}{3n}\left(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

$$\Rightarrow S_{2n} = \frac{b-a}{6n}\left(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{2n-1} + y_{2n}\right)$$

$$y_{2n-1} = \frac{y_{2n-2}+y_{2n}}{2}$$

$$= \frac{b-a}{6n}\left(y_0 + 4\left(\frac{y_0+y_2}{2}\right) + 2y_2 + \cdots\right)$$

$$= \frac{b-a}{6n}\left(3y_0 + 6y_2 + 6y_3 + \cdots + 6y_{2n-2} + 6y_{2n-1} + 3y_{2n}\right)$$

$$= \frac{b-a}{n}\left(\frac{y_0}{2} + y_2 + y_3 + \cdots + y_{2n-1} + \frac{y_n}{2}\right)$$

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

   (a) Find a differential equation expressing the rate at which salt leaves the tank.

   (b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

   (c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.)